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Energy-Dependent Relaxation Times in Electroacoustic Absorption*

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The effect of energy-dependent relaxation times in the theory of the electronic absorption of acoustic waves in nondegenerate semiconductors is considered in this paper. This effect becomes negligible in the two limits of low ($ql \ll 1$) and high ($\omega\tau \gg 1$) frequencies, but for "intermediate" frequencies relative differences as high as 100% have been found in the absorption coefficient for the two cases considered here. These two cases are for $\tau(\epsilon)$ as given (i) by acoustic-phonon scattering and (ii) by ionized-impurity scattering. The comparison is made with the simple case of constant relaxation time.

I. INTRODUCTION

The electronic contribution to the absorption of an ultrasonic wave in semiconductors has been extensively investigated in the last few years. In the earliest theory by Hutson and White,[1] the electronic transport phenomenon is handled through a constant conductivity σ_0, and the diffusion is included through the Einstein relation for static field. These assumptions confine the validity of the theory to the range of frequencies such that $ql \ll 1$, where q is the acoustic wavevector, and l is the mean free path of the carriers. Subsequent theories[2,3] were similar in methodology to the theoretical analysis of the electronic attenuation in metals. These theories have been extended later[4] to include the possibility of nondegenerate distribution functions. However, no theory at present has taken into account the fact that the electronic lifetimes vary greatly with energy in semiconductors, and because of the nondegenerate distribution all electrons are available for interaction with the ultrasonic wave. The purpose of this paper is to include the effects of electronic relaxation times varying with energy in a theory of ultrasonic attenuation appropriate to nondegenerate semiconductors.

Taking the range of relaxation times τ into account appears necessary because of the importance of τ in the previous theories. The attenuation is quite different when $ql<1$, $ql>1$ and $\omega\tau<1$, or $\omega\tau>1$, where $\omega/2\pi$ is the sound frequency. Since the range in relaxation times of the electrons which are strongly coupled to the ultrasonic wave is at least one order of magnitude and varies with the frequency, an appreciable fraction of the electrons may have values of ql or $\omega\tau$ different from what is obtained taking a constant τ_0 as given by the dc conductivity σ_0.

In addition, large changes in ultrasonic attenuation and amplification have been observed in the presence of a longitudinal magnetic field.[5,6] Due to the geometry of the self-consistent field, no such effect is predicted by any existing theory based on constant relaxation times. Good agreement with experiments is achieved, however, by taking into account the effect of the magnetic field on the energy-dependent relaxation time of the carrier.[7]

A detailed analysis of the effect on the electronic attenuation due to the energy dependence of the relaxation time is more complicated than simply averaging the τ-dependent attenuation over the energy. The absorption coefficient α can be easily determined in terms of two frequency-dependent transport coefficients; the conductivity and the diffusion constant. These two coefficients depend in different ways upon the relaxation time and therefore they must be evaluated separately over the distribution function of the electrons.

In the $ql \ll 1$ approximation, as mentioned above, the two transport coefficients are related to each other through the Einstein relation and they have the same value as for a static uniform field. In this case no difference is expected for energy-dependent τ. The limit of very high frequency is the same as the limit of infinite τ for all electrons and again no effect of lifetime range is present. However, for "intermediate" frequencies, where $ql \gtrsim 1$ and $\omega\tau \lesssim 1$ neither one of these two limits are applicable.[8] Under these conditions we expect the dependence of the electronic relaxation time τ on the energy to affect the final result of the theory.

In the evaluation of the transport coefficients, the weighting of τ over the distribution function is characteristic of the dominant electron scattering mechanism. We consider two particular cases where the electron mobility is limited primarily (i) by acoustic-phonon scattering and (ii) by ionized-impurity scattering. The relative difference of the absorption coefficients in these

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[1] A. R. Hutson and D. L. White, J. Appl. Phys. **33**, 40 (1962).

[2] H. N. Spector, Phys. Rev. **125**, 1192 (1962); **127**, 1084 (1962).

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[4] H. N. Spector, Solid State Phys. **19**, 291 (1967).

[5] K. N. Nill and A. L. McWhorter, J. Phys. Soc. Japan Suppl. **21**, 755 (1966).

[6] J. B. Ross, C. Hamaguchi, and R. Bray, Bull. Am. Phys. Soc. **14**, 353 (1969).

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[8] C. Jacoboni and E. W. Prohofsky, J. Appl. Phys. **40**, 454 (1969).

cases reaches values of 100% in the region of "intermediate" frequencies, while at the two limits of low and high frequencies the previous results of the constant τ theories are reproduced, as expected.

In Sec. II we review the general formal theory for the electronic absorption of an acoustic wave in a nondegenerate semiconductor. The theory is linear in the acoustic amplitude and it follows the lines of the well-known paper by Cohen, Harrison, and Harrison.[9] In Secs. III–V we give explicit solutions for the case of τ due to acoustic-phonon scattering (Sec. IV) and due to ionized-impurity scattering (Sec. V); for comparison we include (Sec. III) the results obtained in the case of constant τ, already given in the literature.[8,10] In Sec. VI we present some numerical results for a physical example; these results are analyzed in Sec. VII, where a detailed physical discussion of the whole phenomenon is given.

II. FORMAL THEORY OF ELECTRO-ACOUSTIC ABSORPTION

The model considered in this paper consists of an acoustic wave with velocity field

$$\mathbf{u}(\mathbf{r},t)=\mathbf{u}_0 e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} \tag{1}$$

propagating in a continuous elastic medium with background charge density $n_0 e$ and n_0 carriers per unit volume with charge $-e$ and scalar effective mass m, obeying classical statistics. The elastic medium interacts with the carriers through its charge, a deformation potential, and a piezoelectric polarization. In nondegenerate semiconductors we can neglect the first interaction with respect to the last two. Furthermore, we can assume that the force acting on the electrons due to the acoustic wave is parallel to the wave vector $\mathbf{q}$; this is always true for deformation potential interactions and for non-negligible piezoelectric forces. Consequently, no magnetic field accompanies the wave, and the total force acting on the electrons can be written as

$$\mathbf{F}=-e[\mathbf{E}+(\mathbf{q}\cdot\mathbf{C}\cdot\mathbf{u}/ie\omega)\mathbf{q}], \tag{2}$$

where $\mathbf{E}$ is the self-consistent electric field and $\mathbf{C}$ is the deformation-potential tensor.

By means of Chambers' method,[11] with an electronic relaxation time τ considered constant during each single portion of an electronic path included between two successive collisions, we can write the electronic distribution function to first order in u as[8]

$$f(\mathbf{r},\mathbf{v},t)=f_0\left\{1+\frac{n_1}{n_0}\frac{iv_s}{\omega\tau[v_s+i/(q\tau)-v_z]}-\frac{e\tau}{KT}\frac{iv_sv_z}{\omega\tau[v_s+i/(q\tau)-v_z]}\left(E+\frac{\bar{C}q^2u}{ie\omega}\right)\right\}. \tag{3}$$

Here f_0 is the Maxwell-Boltzmann distribution function; n_1 is the deviation of the carrier density n from its average value n_0; v_s is the velocity of sound; $\mathbf{v}$ is the electron velocity; K is the Boltzmann constant and T is the absolute temperature; $\bar{C}$ is defined in such a way that $\mathbf{q}\cdot\mathbf{C}\cdot\mathbf{u}=q\bar{C}u$; and the direction z has been taken parallel to the wave vector $\mathbf{q}$. In Eq. (3) we have considered the piezoelectric and deformation potential interactions dominant. All other effects such as collision drag or purely electromagnetic forces are considered negligible.

According to Eq. (3) the ac current which is generated by the wave is parallel to z and its value is given by

$$j=\sigma(E+\bar{C}q^2u/ie\omega)-en_1v_sR, \tag{4}$$

where

$$\sigma=\frac{ie^2}{KTq}\int\frac{v_z^2f_0dv}{v_s+i/(q\tau)-v_z} \tag{5}$$

and

$$R=\frac{i}{n_0}\int\frac{v_zf_0dv}{\omega\tau[v_s+i/(q\tau)-v_z]}. \tag{6}$$

The constitutive equation (4) can be solved with the aid of the continuity equation and the Maxwell equations. The resulting self-consistent field is

$$E=\frac{-i\mathfrak{e}qu+i\sigma'\bar{C}q^2u/(e\omega)}{\sigma_0[\sigma'/\sigma_0-i\omega/\omega_r]}, \tag{7}$$

where $\sigma'=\sigma/(1-R)$ and $\omega_r=4\pi\sigma_0/\kappa$ is the dielectric relaxation frequency; σ_0 is the dc conductivity of the sample; κ is the dielectric constant and $\mathfrak{e}$ is the appropriate piezoelectric constant, defined in such a way that the piezoelectric polarization, by assumption parallel to $\mathbf{q}$, is given by

$$P=-(u/v_s)\mathfrak{e}.$$

Equations (4) and (7) can now be used together with the continuity equation to obtain the dissipated power per unit volume and the absorption coefficient α, which then takes the familiar form

$$\alpha=\frac{\omega^2}{\rho v_s{}^3\sigma_0}\left[\mathfrak{e}^2+\left(\frac{\kappa q\bar{C}}{4\pi e}\right)^2\right]\operatorname{Re}\left[\frac{\sigma'}{\sigma_0}-i\frac{\omega}{\omega_r}\right]^{-1}, \tag{8}$$

where ρ is the density of the sample.

This expression gives the electronic acoustic absorption, once the transport coefficients in Eqs. (5) and (6) are evaluated. The angular part of the integrations in those equations can be easily accomplished and the resulting forms are particularly suitable for numerical integration,

$$\frac{\sigma}{\sigma_0}=\frac{2i}{\pi^{1/2}ql_0}\int_0^\infty x\,p\left\{p\ln\frac{p+1}{p-1}-2\right\}e^{-x}dx, \tag{9}$$

$$R=\frac{i}{\sqrt{\pi}}\int_0^\infty\frac{x^{1/2}}{\omega\tau}\left\{p\ln\frac{p+1}{p-1}-2\right\}e^{-x}dx, \tag{10}$$

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[9] M. H. Cohen, M. J. Harrison, and W. A. Harrison, Phys. Rev. **117**, 937 (1960).

[10] H. N. Spector, Phys. Rev. **165**, 562 (1968).

[11] R. G. Chambers, Proc. Phys. Soc. (London) **A65**, 458 (1952); Proc. Roy. Soc. (London) **A238**, 344 (1957).

where

$$x=\epsilon/KT,$$

$$p=p(x)=v_s/v+i/ql,$$

$$l=l(x)=v\tau,$$

$$l_0=v_0\tau_0,$$

$$v_0=(2KT/m)^{1/2},$$

$$\tau_0=\sigma_0 m/n_0e^2,$$

ϵ being the electronic energy.

III. CONSTANT RELAXATION TIME

When τ is equal to a constant τ_0, the integrals in Eqs. (9) and (10) can be exactly evaluated or reduced to well-known special function. The results are a special case of the transport coefficients given in Refs. 8 and 10. We report them here for completeness

$$\sigma/\sigma_0=\sigma^C/\sigma_0=(2/ql_0)g[\pi^{1/2}gw(g)-i], \tag{11}$$

$$R=R^C=(\omega\tau_0)^{-1}[\pi^{1/2}gw(g)-i], \tag{12}$$

where

$$g=(v_s/v_0)(1+i/\omega\tau_0)$$

and w is simply related to the error function[12] (erfc)

$$w(z)=e^{-z^2}\operatorname{erfc}(-iz)=\frac{2iz}{\pi}\int_0^\infty\frac{e^{-t^2dt}}{z^2-t^2},\quad \operatorname{Im}(z)>0. \tag{13}$$

It is of some interest to note that the following relationship exists between the transport coefficients (11) and (12):

$$\sigma^C/\sigma_0=2(v_s/v_0)^2(1+i/\omega\tau_0)R^C. \tag{14}$$

Since R is related to the diffusion constant D by the equation

$$R=-i(q/v_s)D, \tag{15}$$

Eq. (14) reduces to a generalized Einstein relation between the diffusion constant and the mobility μ:

$$\mu=-(eD/KT)(1-i\omega\tau_0). \tag{16}$$

This expression has been obtained by Cohen, Harrison, Harrison[9] for Fermi statistics. The transition to classical statistics can be simply achieved with the substitution $\frac{2}{3}E_F\rightarrow KT$, where E_F is the Fermi energy of the system, due to the particular ways in which the two distributions depend upon the local density of carriers. However, for energy-dependent relaxation times there is no such simple relation between μ and D as τ enters in different ways in the integrals for σ and R.

Substitution of Eqs. (11) and (12) into Eq. (8) gives the absorption coefficient in terms of known quantities for the case of constant relaxation time. The final expression takes the forms given by Hutson and White[1] and by Spector[13] for $ql_0\ll1$ and $ql_0\gg1$, respectively,

$$\alpha^C \underset{ql_0\ll1}{\sim}\left(\frac{4\pi}{\kappa}\right)^2\frac{\sigma_0}{\rho v_s^3}\frac{\mathfrak{e}^2+(\kappa q\bar{C}/4\pi e)^2}{1+(\omega_r/\omega)^2[1+(q/k_d)^2]^2}, \tag{17}$$

$$\alpha^C \underset{ql_0\gg1}{\sim}\left(\frac{4\pi}{\kappa}\right)^2\frac{\sigma_0}{\rho v_s^3}(\tfrac{1}{2}\pi^{1/2}ql_0)\frac{\mathfrak{e}^2+(\kappa q\bar{C}/4\pi e)^2}{(\omega_r/\omega)^2[1+(q/k_d)^2]^2}, \tag{18}$$

where k_d is the reciprocal of the Debye length

$$r_s=(\kappa KT/4\pi n_0e^2)^{1/2}.$$

IV. ACOUSTIC-PHONON SCATTERING

For acoustic-phonon scattering we assume a relaxation time of the form

$$\tau=\tau^A(\epsilon)=\tau_a v_s/v, \tag{19}$$

as given by the deformation potential interaction with thermal acoustic phonons.[14] In this case, it is convenient to first perform the integration in energy and then in the polar angle in Eqs. (5) and (6). After the first integration the resulting expressions are:

$$\frac{\sigma^A}{\sigma_0}=\frac{4i}{\pi^{1/2}ql_0}\int_{-1}^{1}y^2F(y)\,dy \tag{20}$$

and

$$R^A=\frac{2i}{\pi^{1/2}\omega\tau_a}\int_{-1}^{1}yF(y)\,dy, \tag{21}$$

where

$$F(y)=(r/2s)\{1-\tfrac{1}{2}\pi^{1/2}r+r^2-\pi^{1/2}r^3 +i\pi r^4w(-r)+r^4e^{-r^2}E_1(-r^2)\}, \tag{22}$$

and

$$r=r(y)=\frac{s}{i/ql_a-y},$$

$$l_a=v_0\tau_a, \tag{23}$$

$$s=v_s/v_0.$$

w is defined as above, and $E_1(z)$ is the exponential integral function

$$E_1(z)=\int_z^\infty\frac{e^{-t}}{t}\,dt.$$

In the Appendix it is shown that it is a very good approximation for all frequencies to take only the first three terms in Eq. (22) to evaluate σ^A and R^A. When this is done, the following results are obtained after integration in θ:

$$\frac{\sigma^A}{\sigma_0}=\frac{3}{(ql_a)^2}\left\{1-\frac{\arctan(ql_a)}{ql_a}\right\} -3s^2\left\{1-\frac{\arctan(ql_a)}{ql_a}-\frac{(ql_a)^4}{[1+(ql_a)^2]^2}\right\} -\frac{3i(\sqrt{\pi})s}{ql_a}\left\{1-\frac{\arctan(ql_a)}{ql_a}-\frac{1}{2}\frac{(ql_a)^2}{1+(ql_a)^2}\right\}, \tag{24}$$

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[12] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (National Bureau of Standards, Washington D. C., 1964).

[13] H. N. Spector, Phys. Rev. **125**, 1880 (1962).

[14] See, for example, W. Shockley, *Electrons and Holes in Semiconductors* (D. Van Nostrand, Inc., Princeton, N. J., 1950).

$$R^A = -\left\{1 - \frac{\arctan(ql_a)}{ql_a} - \frac{(ql_a)^2}{1+(ql_a)^2}\right\} - \frac{2i}{\pi^{1/2}\omega\tau_a}$$
$$\times\left\{1 - \frac{\arctan(ql_a)}{ql_a} - s^2\frac{(ql_a)^4}{[1+(ql_a)^2]^2}\right\}. \quad (25)$$

Substitution of these results into Eq. (8) gives the acoustic absorption due to electrons interacting with acoustic phonons.

For $ql_a \ll 1$ the resulting expression is equal to Eq. (17) when we take into account that, to obtain the same dc conductivity, τ_a must be taken equal to $\frac{3}{4}\pi^{1/2}\tau_0$. This result is expected, as mentioned in the Introduction.

For $\omega\tau_a \gg 1$ the result is again what is expected for the limiting case of infinite collision time, which is the τ-independent expression given in Eq. (18). However, the condition $ql_a \gg 1$ is not enough to ensure that the infinite collision-time approximation is valid; in fact, in the case of $ql_a \gg 1$ and $\omega\tau_a \ll 1$, the expression for α reduces to

$$\alpha = \alpha^A \underset{ql_a \gg 1}{\sim} \left(\frac{4\pi}{\kappa}\right)^2 \frac{\sigma_0}{\rho v_s{}^3}\left(\frac{\pi^{3/2}}{8}ql_0\right)\frac{\mathrm{e}^2+(\kappa q\bar{C}/4\pi e)^2}{(\omega_r/\omega)^2[1+(q/k_d)^2]^2}, \quad (26)$$

which differs from Eq. (18) by a factor of $\frac{1}{4}\pi$. We return to this point in the final discussion.

V. IONIZED-IMPURITY SCATTERING

When the dominant scattering mechanism for the electrons in the crystal is the Coulomb interaction with ionized impurities, the relaxation time, as given by Brooks and Herring[15] is

$$\tau = \tau^I(\epsilon) = \frac{\tau_i(v/v_0)^3}{\ln(1+\beta)-\beta/(1+\beta)}, \quad (27)$$

where

$$\tau_i = \kappa^2(2m)^{1/2}[(KT)^{3/2}/\pi e^4 N_I], \quad (28)$$

$$\beta = 4\epsilon/\epsilon_s, \quad \epsilon_s = \hbar^2/2mr_s{}^2, \quad (29)$$

and N_I is the density of ionized impurities.

In the evaluation of the dc conductivity σ_0, for $KT/\hbar$ greater than the plasma frequency $\omega_p = (4\pi\sigma_0/\kappa\tau_0)^{1/2}$, the denominator in the right-hand side of Eq. (27) can be taken constant at the value of β for which the electronic energy is $3KT$. When this is done, the normalization to σ_0 provides the value of the constant N_I. On the other hand, when ω is not zero, the condition for the validity of this approximation is more involved because the range of electronic energies which is most important in the determination of the transport coefficients depends upon the value of the frequency. Furthermore, the use of such an approximation is not of much help, if any, in the evaluation of the integrals. Instead, to simplify Eqs. (9) and (10), we note that v_s is much less than v in most of the important range of integration, so that the logarithmic term in these equations can be expanded to first order in $s/x^{1/2}$.

Two problems can arise with this expansion. First, if $\omega\tau^I$ is much larger than unity when $x \approx s^2$, then a resonant contribution to the logarithmic term comes from a range of x in which the expansion is inaccurate. However, when $x \approx s^2$, β is much less than unity in practical cases. For $\beta \ll 1$, the denominator in Eq. (27) reduces to $\frac{1}{2}\beta^2$ so that $\tau^I \approx \tau_i\epsilon_s{}^2/8(KT)^2x^{1/2}$. For $x \approx s^2$, $\omega\tau^I$ is then of the order $ql_ie_s{}^2/8$, where

$$l_i = v_0\tau_i$$

and

$$e_s = \epsilon_s/KT.$$

For the validity of our expansion we therefore assume $ql_i \ll 8/e_s{}^2$. Since in most of the cases of practical interest $e_s \ll 1$, the condition stated above holds even for values of ql_i much larger than unity.

The second problem which can arise with the expansion of the logarithmic term is related to the convergence of the integrals. Even though the expression

$$p\ln(p+1)/(p-1)-2$$

vanishes at $x=0$ for $\tau=\tau^I$, the linear expansion becomes infinite as $x \to 0$. However, the other factors present in the integrals for the transport coefficients ensure their convergence.

After a few simple calculations the resulting expressions are

$$\mathrm{Re}\left(\frac{\sigma^I}{\sigma_0}\right) = \frac{4}{\pi^{1/2}ql_0}\int_0^\infty \left\{\frac{x}{ql^I}\left(1-\frac{\arctan(ql^I)}{ql^I}\right)\right.$$
$$\left.+s^2\left[\arctan(ql^I)-\frac{2ql^I}{1+(ql^I)^2}\right]\right\}e^{-x}dx, \quad (30)$$

$$\mathrm{Im}\left(\frac{\sigma^I}{\sigma_0}\right) = \frac{4s}{\pi^{1/2}ql_0}\int_0^\infty \left\{x^{1/2}\left[2\frac{\arctan(ql^I)}{ql^I}-1\right.\right.$$
$$\left.\left.-\frac{1}{1+(ql^I)^2}\right]+\frac{s^2}{x^{1/2}}\frac{(ql^I)^2}{1+(ql^I)^2}\right\}e^{-x}dx, \quad (31)$$

$$\mathrm{Re}(R^I) = \frac{2}{\pi^{1/2}}\int_0^\infty x^{1/2}$$
$$\times\left[\frac{\arctan(ql^I)}{ql^I}-\frac{1}{1+(ql^I)^2}\right]e^{-x}dx, \quad (32)$$

$$\mathrm{Im}(R^I) = \frac{2}{\pi^{1/2}s}\int_0^\infty \left[-\frac{x}{ql^I}\left(1-\frac{\arctan(ql^I)}{ql^I}\right)\right.$$
$$\left.+s^2\frac{ql^I}{1+(ql^I)^2}\right]e^{-x}dx, \quad (33)$$

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[15] H. Brooks, Advan. Electron. Electron Phys. 7, 85 (1955).

where $l^I = v\tau^I$. Making use of the conditions $ql_i \ll 8e_s$ and $v_s \ll v$ again, the terms proportional to s^2 in Eqs. (30), (31), and (33) may be neglected. If what is left is then substituted in Eq. (8) for α, we obtain

$$\alpha^I = \alpha_> \times \mathcal{R}(ql_i, e_s), \tag{34}$$

where $\alpha_>$ is the expression (18) for α at the limit of infinite τ, and

$$\mathcal{R}(ql_i, e_s) = \frac{1}{\pi^{1/2}} \int_0^\infty \frac{\arctan(ql^I)}{ql^I} x^{1/2} e^{-x} dx \bigg/ \int_0^\infty \left(1 - \frac{\arctan(ql^I)}{ql^I}\right) \frac{x}{ql^I} e^{-x} dx. \tag{35}$$

In deriving Eq. (34) we used the condition $s \ll 1$ and the assumption was made that when $q > k_d$, then $ql_0 \gg s$. The simplification based on this assumption, which is generally valid in all cases of interest, is equivalent to neglecting the term 1 in the denominator of Eq. (17). See also the discussion in Sec. VII.

The ratio $\mathcal{R}$ has been evaluated numerically for several values of ql_i and e_s, and the results are shown in Table I. It is interesting to note that in the region where $ql_i \gg 1$ (but $\omega\tau_i < 1$), $\mathcal{R}$ is close to a constant value, and that if ql is constant in Eq. (3.35), we obtain for $ql \gg 1$ $\mathcal{R} = \frac{1}{4}\pi$ in agreement with the results found in the acoustic phonon case, in which l is indeed a constant.

VI. NUMERICAL EXAMPLE

A numerical example has been fully developed for n-type GaAs at 77°K. To avoid complications with mixed frequency dependences, we considered a transverse wave propagating along a [110] direction, with polarization in the [001] direction. In this case no deformation potential interaction is present ($\bar{C} = 0$) and $\mathfrak{e} = e_{14}$. The following set of parameters were chosen: $\sigma_0 = 2 \times 10^{13}$ sec^{-1}, $n_0 = 1.7 \times 10^{16}$ cm^{-3}, $v_s = 3.35 \times 10^5$ cm/sec, $\kappa = 12.5$, $e_{14} = 6.8 \times 10^4$ esu/cm^2, $m = 0.0667\ m_0$ (m_0 being the mass of the free electron), $\rho = 5.35$ g/cm^3.

TABLE I. Values of the ratio $\mathcal{R}$ [a] for several values of ql_i [a] and e_s.[a]

ql_i \ e_s	0.01	0.05	0.1	0.5	1.0
0.01	156	116	98.0	57.9	41.7
0.02	78.4	58.0	49.2	29.2	21.1
0.04	39.5	29.4	25.0	15.1	11.0
0.07	23.0	17.3	14.8	9.17	6.85
0.1	16.5	12.5	10.8	6.86	5.21
0.2	9.13	7.14	6.28	4.23	3.34
0.4	5.53	4.51	4.06	2.93	2.41
0.7	4.01	3.39	3.10	2.36	1.99
1.0	3.39	2.93	2.71	2.12	1.82
2.0	2.64	2.37	2.23	1.83	1.59
4.0	2.24	2.07	1.98	1.67	1.46
7.0	2.06	1.94	1.86	1.59	1.39
10	1.99	1.89	1.82	1.56	1.36
20	1.92	1.85	1.79	1.51	1.31
40	1.92	1.85	1.80	1.48	1.28
70	1.95	1.88	1.82	1.46	1.27

[a] See Eq. (35).

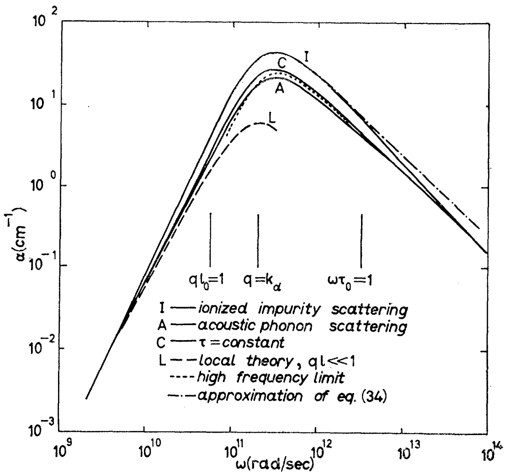

FIG. 1. Absorption coefficient versus frequency in GaAs for the different scattering mechanisms considered in this paper. The acoustic wave is propagating in a [110] direction with polarization in the [001] direction. No deformation potential coupling is present. The parameters used are those given in Sec. VI.

The results are shown in Fig. 1. For $ql_0 \ll 1$ the absorption coefficients obtained for the different collision mechanisms coincide with what is obtained in the local Hutson and White theory. For $\omega\tau_0 \gg 1$ again the result is independent of the scattering because this is equivalent to the result of the infinite τ theory. However, as expected, the different scattering mechanisms give different results in the region of $ql_0 \gtrsim 1$, $\omega\tau_0 \lesssim 1$. It is apparent in Fig. 1 that the curve for the impurity scattering mechanism differs from the limiting curves for low and high frequencies in a wider range of frequencies. Furthermore, the τ = const curve overlaps the limiting curve at $ql \gg 1$ even though $\omega\tau_0$ is still smaller than unity. We discuss the physical reason for this in Sec. VII.

The relative differences between the curves corresponding to different scattering mechanisms become as high at 100%. The ratio α^A/α^C approaches the value $\frac{1}{4}\pi$ as in Eq. (26) at $\omega \sim 3 \times 10^{11}$ rad/sec, where $ql_0 \sim 10$ and $\omega\tau_0 \sim 0.1$, while $\alpha^I/\alpha^C \sim 1.7$ in the same region.

In plotting $\alpha^A(\omega)$ no appreciable difference has been obtained between the exact numerical evaluation of the integrals in Eqs. (9) and (10) and the use of Eqs. (24) and (25). For α^I the difference between the exact numerical calculation and the approximate Eq. (34) is shown in Fig. 1.

VII. DISCUSSION

A. Local Theory, $ql \ll 1$

In this section we discuss in detail the physical content of the calculations developed in Secs. I–VI. For simplicity, consider the case with no deformation

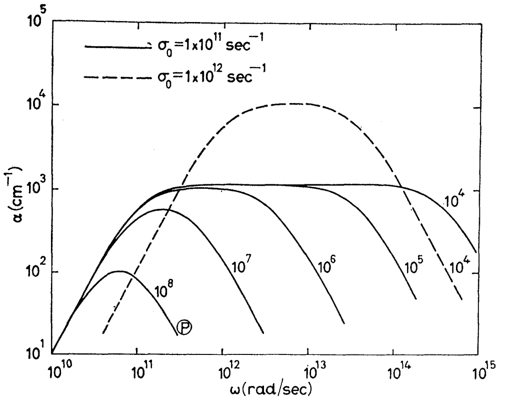

FIG. 2. Absorption coefficient versus frequency as given in Eq. (43) [or Eq. (17)] by the local theory. The numbers on the curves give the values of the product μT in gauss units.

potential coupling and $ql \ll 1$.[1,4,16] The deformation potential, when present, just adds to the constant $\mathfrak{e}^2$ a term proportional to ω^2.

Our geometry is such that no magnetic field accompanies the wave. Therefore, the sum of the true current and the displacement current must be identically zero

$$j - i\omega[(\kappa/4\pi)E + P] = 0. \tag{36}$$

Here j represents the electronic current since the ionic current is assumed to be negligible, and P is the piezoelectric polarization. For $ql \ll 1$ and in absence of diffusion $j = \sigma_0 E$ and Eq. (36) gives

$$E = -(4\pi/\kappa)P = (4\pi/\kappa)(u/\omega)\mathfrak{e}q, \quad \text{for } \omega \gg \omega_r \tag{37a}$$

$$j = i\omega P = -i\mathfrak{e}uq, \quad \text{for } \omega \ll \omega_r. \tag{37b}$$

In the first case no electronic screening is present; the dissipated power is proportional to ω^2. In the second case the screening is complete; j, and therefore E, are proprotional to the time derivative of P, that is, to ω^2. Consequently, the power dissipated is proportional to ω^4. The absorption coefficient α is independent of ω in the case of no screening and proportional to ω^2 when the screening is complete. Making use of Eqs. (37), we have

$$\alpha = \frac{\frac{1}{2}\sigma_0|E|^2}{\frac{1}{2}\rho|u|^2 v_s} = \left(\frac{4\pi}{\kappa}\right)^2 \frac{\mathfrak{e}^2}{\rho v_s^3}\sigma_0 \equiv \alpha_0, \quad \text{for } \omega \gg \omega_r. \tag{38a}$$

$$\alpha = \frac{\frac{1}{2}|j|^2/\sigma_0}{\frac{1}{2}\rho|u|^2 v_s} = \frac{\mathfrak{e}^2\omega^2}{\rho v_s^3 \sigma_0} = \frac{\alpha_0}{(\omega_r/\omega)^2}, \quad \text{for } \omega \ll \omega_r. \tag{38b}$$

If any amount of partial screening is considered, then the full Eq. (36) must be taken and the result is simply for any ω

$$\alpha = \alpha_0 / 1 + (\omega_r/\omega)^2. \tag{39}$$

To include diffusion for $ql \ll 1$, we can use the Einstein relation, together with the continuity condition. The equation $j = \sigma_0 E$ is then replaced by

$$j = \sigma' E, \quad \sigma' = \frac{\sigma_0}{1 + i(ql_0/2s)}. \tag{40}$$

Substitution of Eq. (40) into Eq. (36) leads to results which are exactly like the case with no diffusion with the replacement of σ_0 and κ by

$$\sigma_r' = \operatorname{Re}\sigma' = \frac{\sigma_0}{1 + (ql_0/2s)^2} \tag{41}$$

and

$$\kappa' = \kappa - \frac{4\pi}{\omega}\operatorname{Im}(\sigma') = \kappa\left\{1 + \frac{(\omega_r/\omega)(ql_0/2s)}{1 + (ql_0/2s)^2}\right\}, \tag{42}$$

respectively. We therefore obtain, when the diffusion is taken into account,

$$\alpha = \frac{\alpha_0'}{1 + (\omega_r'/\omega)^2}, \tag{43}$$

where

$$\alpha_0' = (4\pi/\kappa')^2 (\mathfrak{e}^2/\rho v_s^3)\sigma_r' \tag{44}$$

and

$$\omega_r' = (4\pi/\kappa')\sigma_r'. \tag{45}$$

Equation (43) is identical with the Hutson and White result and it can be put into the more familiar form of Eq. (17) by straightforward manipulation. The diffusion becomes effective when the wavelength is comparable with the Debye screening length r_s. In the limit of high frequency ($q \gg k_d$) the diffusion dominates the process (in the $ql \ll 1$ theory). Since no screening is present at this limit, the field is still given by the Eq. (37a). However, the diffusion now essentially reduces the current in phase with the field by a multiplicative factor proportional to $(\sigma_0\omega)^{-2}$ so that the absorption coefficient, which would have been proportional to $\sigma_0\omega^0$ without diffusion becomes proportional to $\sigma_0^{-1}\omega^{-2}$ due to the diffusion.

Figure 2 shows the absorption coefficient versus frequency as given in Eq. (43) or Eq. (17) for several values of the conductivity and the temperature. The values of the parameters have been chosen so as to make apparent the three regions of ω dependence: ω^2 (screening), ω^0 (no screening, no diffusion), and ω^{-2} (diffusion). However, in actual physical cases the diffusion becomes important at frequencies where the screening is still important, so that physical curves always have the characteristic bell shape of the curve indicated by P in Fig. 2.

It is interesting to note that, in spite of the fact that the diffusion does not affect the absorption until $q \approx k_d$, the limit of κ' for $\omega \to 0$ is not $\kappa + 2\pi\sigma_0\tau_0 v_0^2/v_s^2$, which is in general much larger than κ. As a result, the term $(\omega_r'/\omega)^2$ in Eq. (43) can be negligible even in some of the ascending part of the curve of α versus ω, where the screening dominates the phenomenon. In this case E is much larger than it would have been without diffusion.

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[16] A. Rose, RCA Rev. **17**, 98 (1966); **17**, 600 (1966); **18**, 634 (1967).

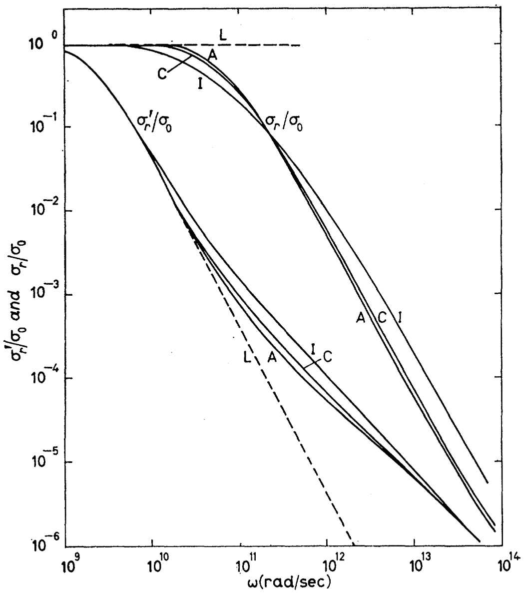

FIG. 3. Real part of the conductivity with and without diffusion, as function of frequency. *I*, *C*, *A*, and *L* have the same meanings as in Fig. 1. The values of the parameters are those given in Sec. VI.

The additional field counteracts the tendency to diffusion, it is out of phase with the current and therefore does not contribute to the absorption. The component of E which is in phase with the current and does contribute to the absorption is independent of diffusion as long as we have complete screening.

B. Intermediate Frequencies, $ql \gtrsim 1$

When the mean free path of the electrons is comparable with or longer than the acoustic wavelength, the current is produced by electrons coming from different regions of the wave. The correlation between the field and the current is decreased and therefore the conductivity is decreased. We refer to this effect as the ql effect. Equation (43) is still valid if σ' is evaluated with exact transport coefficients obtained in Secs. I–VI.

In the case of total screening, the decrease in conductivity due to the ql effect requires a larger field to maintain electrical neutrality and the total absorption is increased. More formally we can say that a decrease in σ produces an increase in α in the case of total screening because Eq. (38b) contains σ in the denominator. This explains the larger values of α for the exact solutions as compared with the local theory in the ascending part of α versus ω in Fig. 1.

Now it can be seen why the impurity scattering curve in Fig. 1 is higher than the constant τ curve, and why the acoustic-phonon curve is lowest in the ascending part of the α curve. For, when ql reaches values comparable with unity, electrons with higher velocity average their contributions to zero over several wavelengths ($v_0 \gg v_s$). The effective average electron energy is thus lower than for the case of dc conductivity. Going to lower energies $\tau^I(\epsilon)$ decreases, and therefore the decrease in σ due to the ql effect is enhanced. On the other hand, $\tau^A(\epsilon)$ increases for decreasing electron energies and the ql effect is less effective in reducing the conductivity.

Since $v_s \ll v_0$, from Eqs. (41) and (42) it is apparent that, unless ω_r is exceedingly large, the second term in the denominator of Eq. (43) is negligible when $ql \gtrsim 1$. Furthermore, in the limit of high frequency κ' approaches κ. Therefore for $ql \gtrsim 1$ the behavior of α is determined by the behavior of σ_r:

$$\sigma_r' = [(1-R_r)\sigma_r - R_i\sigma_i]/[(1+R_r)^2 + R_i^2], \tag{46}$$

where σ_r, σ_i, R_r, R_i are the real and imaginary parts of σ and R. Figure 3 shows plots of σ_r and σ_r' as function of frequency. It is apparent there that the first effect of ql approaching one is the decrease in σ_r and the corresponding increase in α discussed above.

At somewhat higher values of the frequency the picture is more complicated as shown by the crossing of the curves for σ_r versus ω in Fig. 3. To understand this crossing we have to consider that the change in τ with energy alters ql itself. In the case of acoustic phonon scattering l is a constant so that ql increases with frequency only because of the q factor. In the case of constant τ, besides the increase in q with increasing frequency, there is a decrease in l due to the lowering of the effective electronic energy seen before. This decrease in l is still more important in the impurity scattering case in which the relaxation time also decreases with decreasing electronic energy. At high enough frequency this relative reduction of ql becomes so important that the ql effect in the I case is actually less effective in reducing σ than in the C case, and in the C case is less effective than in the A case. Therefore, the simple dependence of σ upon the scattering mechanism which we have seen before is reversed. However, at these frequencies the electronic bunching is essentially reduced and so is the diffusion current. Therefore, σ' and σ become similar in meaning, contrary to the local theory, and no crossing appears in σ_r'. In other words, at these frequencies the screening is no longer present and higher σ produce higher absorption so that no crossing appears in α.

C. High Frequencies and the High-Frequency Limit

In the limit of $ql \gg 1$ the picture of the phenomenon is completely different: The electronic contribution to the absorption comes from the resonant interaction of the carriers traveling along with the wave with $v_z = v_s$. This can be seen directly from Eq. (5), which in the case of constant τ gives

$$\sigma_r = \frac{2\sigma_0}{\pi^{1/2}\omega\tau_0} \int_{-\infty}^{\infty} \frac{e^{-\xi^2}\xi^2 d\xi}{(1-\xi/s)^2 + (1/\omega\tau_0)^2}, \tag{47}$$

where ξ represents v_z/v_0. The width of the resonance at $\xi=s$, i.e., at $v_z=v_s$, is $2/(ql_0)$. At the peak the integrand is approximately $(s\omega\tau_0)^2$, for $s\ll 1$. Therefore the contribution to σ_r due to the resonant electrons is approximately $4s^2\sigma_0/(\pi^{1/2}ql_0)$. At the limit of infinitely large $\omega\tau$ the exact value of σ_r is in fact

$$\sigma_r(\infty)=(2\pi^{1/2}/ql_0)s^2\sigma_0. \tag{48}$$

This limiting expression, which is independent of τ_0, is obtained both for $\tau=\tau_0$ and for $\tau=\tau^A(\epsilon)$, and we can expect to be valid for any form of $\tau(\epsilon)$ because it corresponds to the limit of infinite relaxation times.

The nonresonant part of the integrand is essentially different from zero in a range of ξ of the order of unity and it has an average value on this interval of the order of s^2 for $ql_0\gg 1$, so that the contribution of the nonresonant electrons to σ_r is of the order of $s^2\sigma_0/(\omega\tau_0)^2$. Therefore, the condition for σ_r to be determined essentially by resonant electrons is that $\omega\tau_0\gg 1/s$. This condition is not only stronger than $ql\gg 1$, but it is even stronger than $\omega\tau_0\gg 1$. In Fig. 3 it can be actually seen that the various σ_r do not reach the limiting curve even at $\omega\tau_0>1$.

However, in the evaluation of σ_r' the terms containing $1/(\omega\tau_0 s)$ cancel in the numerator of Eq. (46). A physical interpretation of this fact can be given by saying that the diffusion tends to destroy residual bunching due to finite collision time.

In the expansion in $1/(ql)$ the next correction over the high-frequency limit is of the order of $1/(ql)\times 1/(\omega\tau s)=(1/\tau\omega)^2$. There is again a cancellation of these terms in the evaluation of σ_r'. This time, however, the cancellation, which occurs between numerator and denominator in Eq. (46), is exact only for the case of constant τ where no different weighting occurs in the evaluation of σ and R. In the case of $\tau^I(\epsilon)$ and of $\tau^A(\epsilon)$ the cancelation is not complete. When $\omega\tau_0\ll 1$ (but $ql_0\gg 1$), the corrections of order $(1/\omega\tau_0)^2$ are dominant and the exact form of the corrections depends on the weighting of $\tau(\epsilon)$ in σ and R. This is the origin of the constant factors seen in this range of frequency in Secs. IV and V.

It is clear now that the high-frequency limit is reached for $\omega\tau\gg 1$ and not for $ql\gg 1$ in the general case of energy-dependent relaxation times. In Fig. 3 it can be seen that the various σ_r' do join at $\omega\tau_0>1$. Only in the case of constant τ do the corrections due to finite $\omega\tau$ cancel exactly and the $\omega\tau\gg 1$ limit is already reached when $ql\gg 1$. The curve I in Fig. 1 approaches the high-frequency limit at higher frequencies than the curve A because the $(1/\omega\tau)^2$ corrections are larger in the former case due to the decrease in τ^I and the increase in τ^A at lower electron energy.

In the limit of $\omega\tau\gg 1$ the denominator of Eq. (46) reduces to one. The numerator reduces to $\sigma_r(\infty)$ as given in Eq. (48). Then σ_r' becomes equal to σ_r and, contrary to the local theory, the diffusion has no effect in the high-frequency limit. As we have seen the diffusion does, however, affect the frequency at which the limiting case is achieved.

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APPENDIX

In this Appendix, we show that in the evaluation of the transport coefficients σ^A and R^A treated in Sec. IV, it is sufficient to keep only the first three terms in the expression (22) for the function $F(y)$.

First, we note that the special functions E_1 and w which appear in Eq. (22) can be put into the form[12]

$$E_1(z)=-\gamma_E-\ln z-\sum_{n=1}^{\infty}\frac{(-1)^n z^n}{nn!},$$

$$w(z)=\sum_{n=0}^{\infty}\frac{(iz)^n}{\Gamma(1+\frac{1}{2}n)},$$

where γ_E represents the Euler constant.

We then note that the maximum value of $|r|$, as given in Eq. (23), is $\omega\tau_a$. Therefore, for $\omega\tau_a\ll 1$ it is sufficient to keep the terms with the lowest powers in r in Eq. (22).

If $\omega\tau_a$ is not much less than one, then $ql_a\gg 1$ because $v_s\ll v_0$. Let us take a general term of the form

$$F_n=a_n r^n.$$

Substitution of F_n into the integral in Eq. (20) for σ^A leads to a straightforward integration, and the result has the form

$$a_n s^n\left\{\frac{\delta^2}{n-1}\left[\frac{1}{(i\delta+1)^{n-1}}-\frac{1}{(i\delta-1)^{n-1}}\right]-\frac{1}{n-3}\left[\frac{1}{(i\delta+1)^{n-3}}-\frac{1}{(i\delta-1)^{n-3}}\right]+\frac{2i\delta}{n-2}\left[\frac{1}{(i\delta+1)^{n-2}}-\frac{1}{(i\delta-1)^{n-2}}\right]\right\},$$

where $\delta=1/ql_a$. For $\delta\ll 1$ these terms become small at least as s^n. It is therefore a very good approximation to keep only the first three terms in the expression (22) to evaluate σ^A. An essentially identical argument holds for the evaluation of R^A.